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NOTES IN THE THEORY  
OF DYNAMIC PROGRAMMING - III:  
EQUIPMENT REPLACEMENT POLICY

by

Richard Bellman

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SUMMARY

~~In this paper we apply~~ the functional equation  
technique of the theory of dynamic programming  
to the theory of equipment replacement.

NOTES IN THE THEORY OF DYNAMIC PROGRAMMING - III  
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§1. Introduction

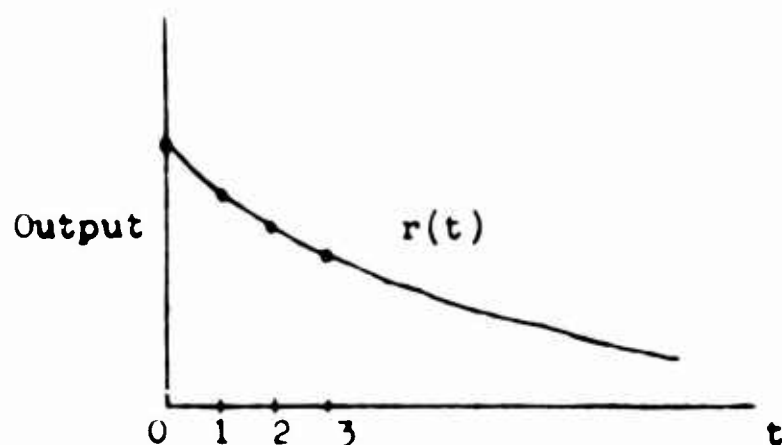
A problem of interest in industrial economics is that of determining the optimal procedure for the replacement of old equipment by new. This problem has been treated by Alchian in a very interesting monograph, [1]. In this paper we wish to show how the problem may be approached by the functional equation technique of the theory of dynamic programming.\* We shall first consider the simple case in which there is no technological improvement, and show that this problem may be resolved very easily. Then we shall consider the more difficult and important case where there is technological improvement, and derive a functional equation for the solution of this problem. A more detailed discussion of this equation will be given in a further communication.

§2. The Process

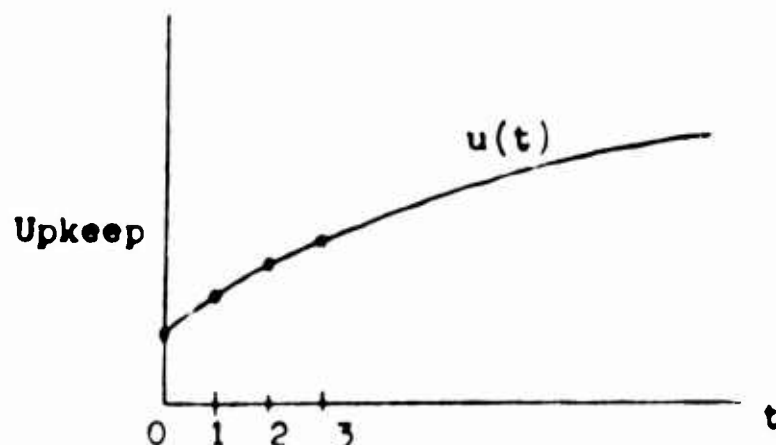
Let us assume that we have a piece of equipment with the following operating characteristics:

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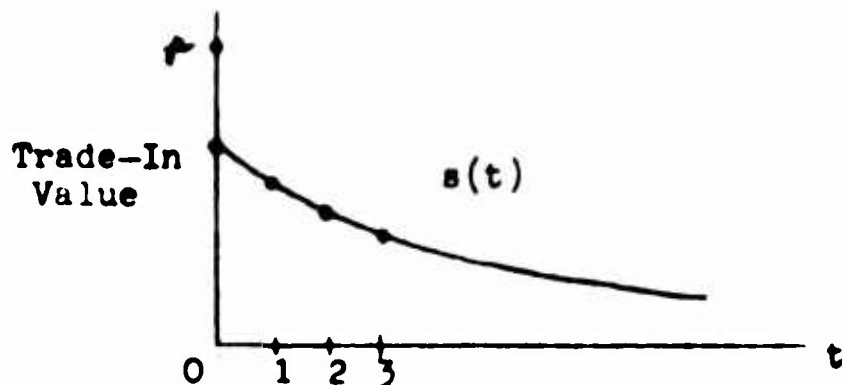
- \* An exposition of the theory and an extensive bibliography may be found in [2].



$t$  = Age of Machine



Here  $t$  is the age of the machine measured in appropriate units. Furthermore, let us assume that the trade-in value of the machine as a function of time has the form



where  $p$  is the purchase price of a new machine.

Given this information, the problem is to determine the age at which the machine should be replaced. We shall assume that we are considering an unbounded process where we keep using machines and replacing machines and so forth. Furthermore, let us assume that machines can only be replaced at times  $t = 1, 2, \dots$ , and that there is immediate delivery.

In order to keep the over-all return finite, let us discount the return one stage ahead by a factor  $a$ , where  $0 < a < 1$ .

Under these conditions, we wish to determine the replacement policy which maximizes the over-all discounted return.

### §3. Mathematical Formulation

It is clear that the only state variable is  $t$ , the age of the machine. Let us then define

- (1)  $f(t)$  = overall return from a machine of age  $t$  employing an optimal replacement policy.

At each time  $t$ , we have one of two courses of action open. We may either keep the machine for another time period, or we may purchase a new machine. In the first case, we see that  $f(t)$  satisfies the equation

$$(2) \quad f_K(t) = r(t) - u(t) + af(t+1) ,$$

and in the second case

$$(3) \quad f_P(t) = s(t) - p + r(0) - u(0) + af(1) .$$

Hence the functional equation for  $f(t)$  is

$$(4) \quad f(t) = \text{Max} \begin{bmatrix} K: & r(t) - u(t) + af(t+1) \\ P: & s(t) - p + r(0) - u(0) + af(1) \end{bmatrix}$$

### §4. Solution

An optimal policy will have the form: keep a new machine until it is  $T$  years old and then purchase a new one. This yields the following system of equations

$$\begin{aligned}
 (1) \quad & f(0) = n(0) + af(1) \\
 & f(1) = n(1) + af(2) \quad , \\
 & f(T-1) = n(T-1) + af(T) \quad , \\
 & f(T) = s(T) - p + n(0) + af(1) \quad ,
 \end{aligned}$$

where we have set  $n(t) = r(t) - u(t)$ . It is clear from (3.4) that  $T \neq 0$ , since  $p > s(0)$ . Solving for  $f(1)$  recurrently, we obtain

$$\begin{aligned}
 (2) \quad & f(1) = n(1) + a(n(2) + af(3)) = n(1) + an(2) + a^2f(3) \\
 & = n(1) + an(2) + \dots + a^{T-2}n(T-1) + a^{T-1}f(T) \\
 & = [n(1) + an(2) + \dots + a^{T-2}n(T-1)] \\
 & = a^{T-1} [s(T) - p + n(0) + af(1)] \quad .
 \end{aligned}$$

Hence

$$(3) \quad f(1) = \frac{[n(1) + an(2) + \dots + a^{T-2}n(T-1) + n(0)a^{T-1}] + a^{T-1}[s(T) - p]}{1 - a^T}$$

Since  $T$  is to be chosen to maximize  $f(0)$ , and hence  $f(1)$ , we see that the optimal value of  $T$  is furnished by the value of  $T$  which yields the absolute maximum of the right side of (3) above for  $T = 1, 2, \dots$ . Let us call this maximum  $T$ .

This is a very simple computation to perform given the preceding curves.

## §5. Over-Age Machines

Actually we do not have the complete solution to the problem, since we do not know what to do with an over-age

machine, one whose age is greater than  $T$ . It is not clear that in this case the optimal policy involves purchasing a new machine immediately, and, as a matter of fact, it is not unrestrictedly true. To determine the optimal procedure, we use the relation in (3.4) for  $t \geq T + 1$ . There will then be a second critical age  $T_2$ , determined very much the same way as above, at which we purchase a new machine starting with this over-age machine. It is clear that by suitably altering  $n(t)$  and  $s(t)$  for large values of  $t$ , we can have as many switchover points as we wish.

#### §6. Technological Improvement

Let us now consider the more realistic situation where improved operating techniques will increase the future return from the same machine, and new machines will be developed. There are now two state variables,  $t$ , the age of the machine, and  $\tau$ , absolute time.

Let us define as above

- (1)  $f(\tau, t)$  = overall return obtained from a machine of age  $t$  at time  $\tau$ , using an optimal replacement policy.

The functional equation is

$$(2) \quad f(\tau, t) = \text{Max} \left[ \begin{array}{l} K: \quad r(\tau, t) - u(\tau, t) + af(\tau+1, t+1) \\ P: \quad s(\tau, t) - p + r(\tau, 0) - u(\tau, 0) \\ \quad \quad \quad + af(\tau+1, 1) \end{array} \right]$$



where the graphs given in §1 typify families of such graphs dependent upon the absolute time. This equation is more difficult to treat than the one above. Here the method of successive approximations based upon an initial policy space approximation will work well. We shall discuss this in a further communication.

#### BIBLIOGRAPHY

1. A. Alchian, "Economic Replacement Policy", RAND Report No. R-224.
2. R. Bellman, "The Theory of Dynamic Programming", Bull. Amer. Math. Soc., Vol. 60 (1954), pp. 503-516.